Baby problem:

$$Z[J] = \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4 + Jq} \qquad (1)$$

$\longrightarrow$ diagramatic expansion

Child problem:

$$Z[J] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dq_1\, dq_2 \cdots dq_N\, e^{-\frac{1}{2}q.A.q - (\frac{\lambda}{4!})q^4 + J.q} \qquad (2)$$

with $q^4 := \sum_i q_i^4$

$$\longrightarrow Z[J] = \left[\frac{(2\pi)^N}{\det A}\right]^{\frac{1}{2}} e^{-\frac{\lambda}{4!}\sum_i \left(\frac{\partial}{\partial J_i}\right)^4} e^{\frac{1}{2}J.A^{-1}.J}$$

Alternatively, we expand in powers of J:

$$Z[J] = \sum_{s=0}^{\infty} \frac{1}{s!} J_{i_1}\cdots J_{i_s} \int_{-\infty}^{\infty}\left(\prod_\ell dq_\ell\right) e^{-\frac{1}{2}q.A.q - \frac{\lambda}{4!}q^4}\{q_i\}$$

$$= Z[0,0] \sum_{s=0}^{\infty} \frac{1}{s!} J_{i_1}\cdots J_{i_s} G^{(s)}_{i_1\ldots i_s}$$

Let's evaluate the 2-point function:

$$G^{(2)}_{ij}(\lambda=0) = \left[\int_{-\infty}^{\infty}\left(\prod_\ell dq_\ell\right) e^{-\frac{1}{2}q.A.q}\, q_i q_j\right] / Z[0,0]$$

$$= (A^{-1})_{ij}$$

$\rightarrow$ $(A^{-1})_{ij}$ describes the amplitude of "propagation" from i to j

Let us next evaluate the 4-point fkt.:

$$G^{(4)}_{ijkl} = \int_{-\infty}^{\infty} \left(\prod_m dq_m \right) e^{-\frac{1}{2} q \cdot A \cdot q} q_i q_j q_k q_l$$

$$\times \left[1 - \frac{\lambda}{4!} \sum_n q_n^4 + \mathcal{O}(\lambda^2) \right] / Z[0,0]$$

"Wick contractions"

$$= (A^{-1})_{ij} (A^{-1})_{kl} + (A^{-1})_{ik} (A^{-1})_{jl} + (A^{-1})_{il} (A^{-1})_{jk}$$

$$- \lambda \sum_n (A^{-1})_{in} (A^{-1})_{jn} (A^{-1})_{kn} (A^{-1})_{ln} + \mathcal{O}(\lambda^2)$$

- First three terms describe one excitation propagating from i to j and another from k to l + permutations
- Order λ term: four excitations, propagating from i to n, from j to n, from k to n, from l to n, form a "vertex" with amplitude λ

Perturbative Field Theory

$$Z[J] = \int \mathcal{D}\varphi\, e^{i\int d^4x \left(\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] - \left(\frac{\lambda}{4!}\right)\varphi^4 + J\varphi\right)} \quad (3)$$

has same form as integrals (1), (2)

Thus

$$Z[J] = Z[0,0]\, e^{-\left(\frac{i}{4!}\right)\lambda \int d^4w\, [\delta/i\delta J(w)]^4} \times \int \mathcal{D}\varphi\, e^{i\int d^4x \left(\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] + J\varphi\right)}$$

$$= Z[0,0]\, e^{-\left(\frac{i}{4!}\right)\lambda \int d^4w\, [\delta/i\delta J(w)]^4} \times e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)}$$

where

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon}$$

propagation in "absence" of interaction

$$\longrightarrow Z[J] = Z[0,0] \sum_{s=0}^{\infty} \frac{1}{s!} J(x_1) \cdots J(x_s)\, G^{(s)}(x_1, \ldots, x_s)$$

$$= \sum_{s=0}^{\infty} \frac{1}{s!} J(x_1) \cdots J(x_s) \int \mathcal{D}\varphi\, e^{i\int d^4x \left(\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!}\varphi^4\right)} \times \varphi(x_1) \cdots \varphi(x_s)$$

$\rightarrow$ 2-point Green's function:

$$G(x_1, x_2) = \frac{1}{Z[0,0]} \int \mathcal{D}\varphi \, e^{i\int d^4x \left(\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!}\varphi^4\right)} \times \varphi(x_1)\varphi(x_2)$$

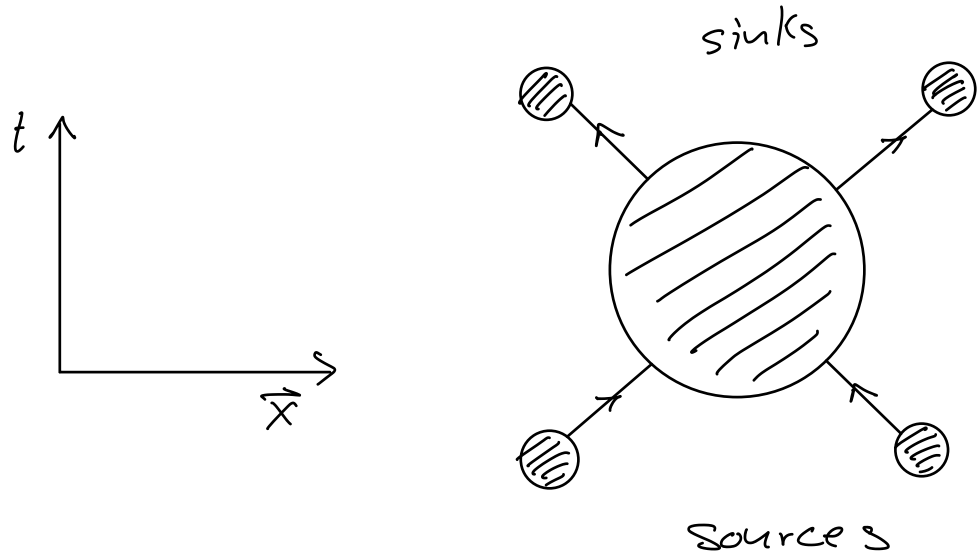
Propagation in presence of interaction

4-point Green's function:

$$G(x_1, x_2, x_3, x_4) = \frac{1}{Z[0,0]} \int \mathcal{D}\varphi \, e^{i\int d^4x \left(\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!}\varphi^4\right)} \times \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)$$

describes scattering of particles

Collision between particles



To order λ, we thus have to compute

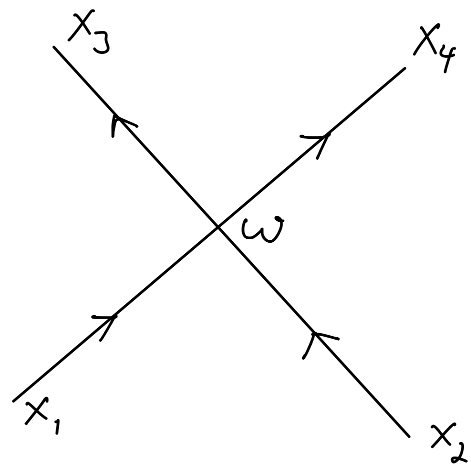
$$\frac{1}{Z[0,0]}\left(-\frac{i\lambda}{4!}\right)\int d^4w \int \mathcal{D}\varphi\, e^{i\int d^4x\, \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]}$$

$$\times\, \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\varphi(w)^4$$

$\longrightarrow$ Wick contraction gives

$$(-i\lambda)\int d^4w\, D(x_1-w)D(x_2-w)D(x_3-w)D(x_4-w)$$

diagramatically:



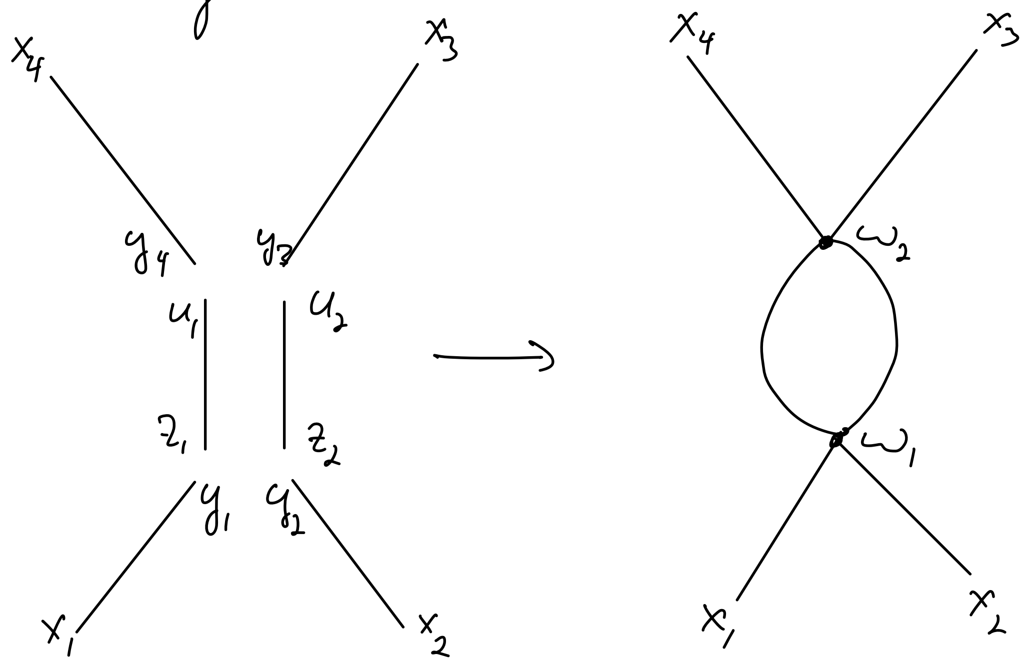
More generally: operator $e^{-\frac{i}{4!}\lambda\int d^4w\,[\delta/\delta J(w)]^4}$ acting on

$$Z[J,\lambda=0) = e^{-\frac{i}{2}\int d^4x\,d^4y\,J(x)D(x-y)J(y)}$$

$\rightarrow$ at second order:

$$\sim \lambda^2 \int d^4w_1 \int d^4w_2 \left[\frac{\delta}{\delta J(w_1)}\right]^4 \left[\frac{\delta}{\delta J(w_2)}\right]^4$$

producing:



<u>Wick's theorem:</u>

Generalizing the above, we obtain

$$Z[J] = \frac{1}{Z[0]} e^{-\int dx\,\mathcal{L}_{int}\left(\frac{\delta}{\delta J(x)}\right)} e^{-\frac{i}{2}\int d^4x\,d^4y\,J(x)D(x-y)J(y)}$$

All Green functions can be obtained as

$$G^{(N)}(x_1, x_2, \dots, x_N) = \frac{\delta^N Z[J]}{\delta J(x_1)\cdots\delta J(x_N)}\bigg|_{J=0}$$

→ for $G^{(N)}$ to not vanish, all derivative terms must group in pairs, in all possible ways!

→ For each pair there will be a factor of $D(x-y)$, where x and y are the coordinates of the J's in the pair.

In operator language:

define $\langle 0| T[\varphi(x_1)\cdots\varphi(x_N)]|0\rangle_0$ ← no interaction

$$:= \int \mathcal{D}\varphi\ \varphi(x_1)\cdots\varphi(x_N)\, e^{i\int d^4x\, \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]}$$

$$= \left(\frac{\delta^N}{\delta J(x_1)\cdots\delta J(x_N)}\int \mathcal{D}\varphi\, e^{i\int d^4x\,\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] + J\varphi}\right)\bigg|_{J=0}$$

$$= \left(\frac{\delta^N}{\delta J(x_1)\cdots\delta J(x_N)}\underbrace{e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)}}_{=:\, Z^0[J]}\right)\bigg|_{J=0}$$

and $\varphi(x)\varphi(y) = D_F(x-y) = -iD(x-y)$

Then $\langle 0| T[\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\cdots\varphi(x_N)]|0\rangle_0$

$$= \langle 0|\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\cdots\cdots\varphi(x_N)|0\rangle_0$$

$$+ \langle 0|\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\cdots\varphi(x_N)|0\rangle_0$$

$$+ \cdots$$

$$+ \langle 0|\varphi(x_1)\,\varphi(x_2)\cdots\cdots\varphi(x_N)|0\rangle_0$$

Example: Two-point function in φ^4-th

$$\mathcal{L}_{int} = \frac{\lambda}{4!}\varphi^4$$

$$\rightarrow Z[J] = \frac{1}{Z[0]}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\lambda}{4!}\right)^n$$

$$\times \int dx_1\cdots dx_n\left[\frac{\delta}{\delta J(x_1)}\right]^4\left[\frac{\delta}{\delta J(x_2)}\right]^4\cdots\left[\frac{\delta}{\delta J(x_n)}\right]^4 Z^0[J]$$

$\rightarrow$ To 0-th order in λ:

$$\langle 0|T[\varphi(x)\varphi(y)]|0\rangle = D_F(x-y)$$

$\rightarrow$ To 1-th order in λ:

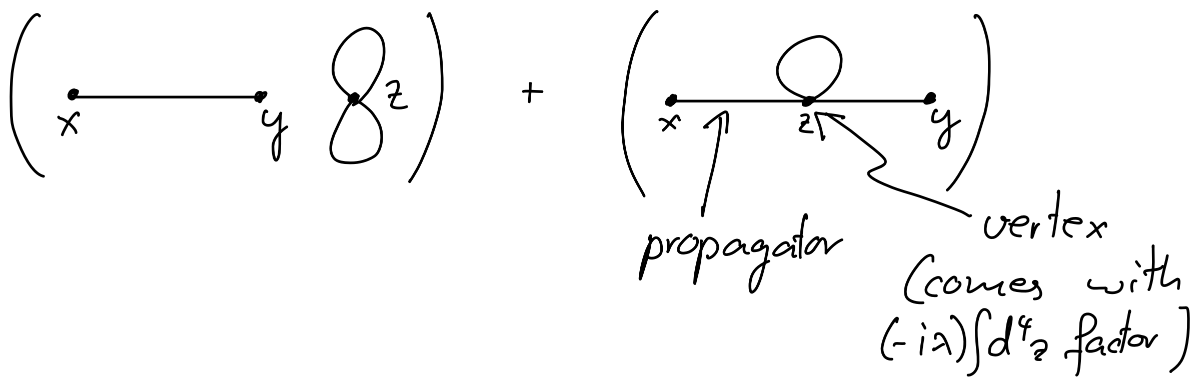
$$\langle 0|T\left[\varphi(x)\varphi(y)(-i)\int d^4z\,\frac{\lambda}{4!}\varphi(z)^4\right]|0\rangle$$

$$= 3\cdot\left(-\frac{i\lambda}{4!}\right)D_F(x-y)\int d^4z\,D_F(z-z)D_F(z-z)$$

$$+ 12\cdot\left(-\frac{i\lambda}{4!}\right)\int d^4z\,D_F(x-z)D_F(y-z)D_F(z-z)$$

- the factor 3 comes from 3 ways to contract $\varphi(z)$'s with each other
- the factor 12 comes from $4 \cdot 3$ ways to contract $\varphi(x)$ and $\varphi(y)$ with $\varphi(z)$'s

$\longrightarrow$ in terms of Feynman diagrams:



Let's now try the order λ^3 term:

$$\langle 0|\varphi(x)\varphi(y)\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\,\varphi\varphi\varphi\varphi\int d^4w\,\varphi\varphi\varphi\varphi\int d^4u\,\varphi\varphi\varphi\varphi|0\rangle$$

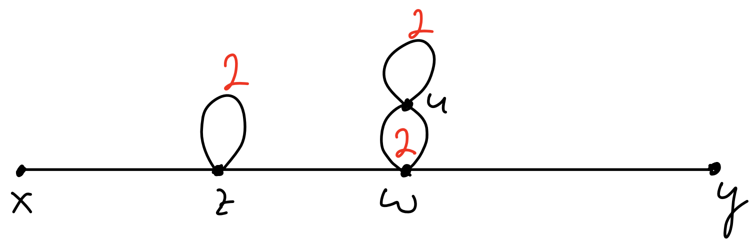
$$= \frac{1}{3!}\left(-\frac{i\lambda}{4!}\right)^3\int d^4z\,d^4w\,d^4u\; D_F(x-z)D_F(z-z)D_F(z-w)$$
$$\times D_F(w-y)D_F^2(w-u)D_F(u-u)$$

$\longrightarrow$ number of "different" contractions producing same expression:

$$\underbrace{3!}_{\substack{\text{interchange}\\ \text{of vertices}}} \times \underbrace{4\cdot 3}_{\substack{\text{placement}\\ \text{of contractions}\\ \text{into } z \text{ vertex}}} \times \underbrace{4\cdot 3\cdot 2}_{\substack{\text{placement}\\ \text{of contractions}\\ \text{into } w \text{ vertex}}} \times \underbrace{4\cdot 3}_{\substack{\text{placement}\\ \text{of contractions}\\ \text{into } u\\ \text{vertex}}} \times \underbrace{\frac{1}{2}}_{\substack{\text{interchange}\\ \text{of}\\ u\text{-}w\\ \text{contractions}}}$$

$= 10,368$

Corresponding Feynman diagram:



If we neglect the $\frac{1}{4!}$ factors (placements into vertex) and the $\frac{1}{3!}$ factors (interchange of vertices cancels these), we get

$$\frac{3!\,(4!)^3}{10,368} = \underset{\substack{\shortparallel \\ 2\cdot 2\cdot 2}}{8} \longrightarrow \text{"symmetry factor" of diagram}$$